(7 pages)

Reg. No. : ....

Code No. : 5852 **Sub. Code : PMAM 41**

M.Sc. (CBCS) DEGREE EXAMINATION,
APRIL 2020.

Fourth Semester

Mathematics — Core

FUNCTIONAL ANALYSIS

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. $\sum_{i=1}^{n} |x_i \, y_i| \leq \|x\|_p \, \|y\|_q$ is known as the

   (a) Minkowski's inequality

   (b) Holder's inequality

   (c) Triangle inequality

   (d) Schwartz's inequality

2. The conjugate space of l_∞^n is

(a) l_∞^n (b) l_1^∞

(c) l_1^n (d) l_∞

3. Which one of the following is not reflexive?

(a) l_3 (b) C_0

(c) L_4 (d) l_2^6

4. If X is a compact Hausdorff space then $C(X)$ is reflexive if and only if

(a) X is an infinite set

(b) X is an uncountable set

(c) X is a finite set

(d) X is a singleton set

5. In a Hilbert space $\langle x, iy + 4z \rangle$ is

(a) $i\langle x, y\rangle + 4\langle x, z\rangle$ (b) $-i\langle x, y\rangle - 4\langle x, z\rangle$

(c) $-i\langle x, y\rangle + 4\langle x, z\rangle$ (d) $\langle x, y\rangle + 4\langle x, z\rangle$

6. In a Hilbert space H, which one of the following is not true (Here S is a nonempty subset of H)

(a) $\{0\}^\perp = H$

(b) $S \cap S^\perp \subseteq \{0\}$

(c) $S_1 \subseteq S_2 \Rightarrow S_1^\perp \subseteq S_2^\perp$

(d) $S^\perp$ is a closed linear subspace of H

7. The value of $\int_0^{2\pi} e^{i4x} e^{-7x} dx$ is

(a) -3 (b) 11

(c) 0 (d) 2π

8. An operator T on H is normal if and only if

(a) $\|T^* x\| = \|Tx\|$ for every x

(b) $\|Tx\| = \|x\|$ for all x

(c) $(Tx, Ty) = (x, y)$ for all x and y

(d) (Tx, x) is real for all x

9. A closed linear subspace M of H reduces an operator T if and only if M is invariant under

(a) T (b) T^*

(c) either T or T^* (d) both T and T^*

10. An operators T on H is an isometric isomorphism of H onto itself if and only if

(a) T is unitary

(b) T is normal

(c) T is self adjoint

(d) T is a positive operator

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Define a Banach space with an example. Prove that the addition and the scalar multiplication are jointly continuous in a Banach space.

Or

(b) If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, prove that there exists a functional f_0 in N^* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$.

12. (a) If B and B' are Banach spaces, and if T is a linear transformation of B into B', prove that T is continuous if its graph is closed.

Or

(b) If B is a Banach space, prove that B is reflexive if and only if B^* is reflexive.

13. (a) Prove that a non-empty subset X of a normed linear space N is bounded $\Leftrightarrow f(X)$ is a bounded set of numbers for each f in N^*.

Or

(b) State and prove Schwarz inequality in a Hilbert space.

14. (a) Prove that a Hilbert space H is separable $\Leftrightarrow$ every orthonormal set in H is countable.

Or

(b) Prove that an operator T on H is self-adjoint $\Leftrightarrow$ (Tx, x) is real for all x.

15. (a) If T is an operator on H, prove that T is normal $\Leftrightarrow$ its real and imaginary parts commute.

Or

(b) If P is the projection on a closed linear subspace M of H, prove that M is invariant under an operator $T \Leftrightarrow TP = PTP$.

PART C — $(5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Let M be a closed linear subspace of a normed linear space N. If the norm of a coset $x + m$ in the quotient space N/M is defined by

$\|x + M\| = \inf \{\|x + M\| / m \in M\}$, prove that N/M is a normed linear space. Also show that N/M is a Banach space if N is a Banach space.

Or

(b) Let M be a linear subspace of a normed linear space N and let f be a functional defined on M. If x_0 is a vector not in - M and it $M_0 = M + [x_0]$ is the linear subspace spanned by M and x_0, prove that f can be extended to a functional f_0 defined on M_0 such that $\|f_0\| = \|f\|$.

17. (a) Prove that $x \to F_x$ is a norm preserving mapping of N into $N^{*}x$ where F_x is defined by $F_x(f) = f(x) \forall f \in N^{*}$. Also show that the mapping $x \to F_x$ is linear and an isomeric isomorphism of N into N^{*}.

Or

(b) If B and B' are Banach spaces, and it T is a continuous linear transformation of B onto B', prove that the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B'.

18. (a) State and prove the uniform boundedness theorem.

Or

(b) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

19. (a) Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H. Prove that the following are equivalent :

(i) $\{e_i\}$ is complete

(ii) $x \perp \{e_i\} \Rightarrow x = 0$

(iii) If x is an arbitrary vector in H then $x = \sum (x, e_i) e_i$

(iv) If x is an arbitrary vector in H, then $\|x\|^2 = \sum |(x, e_i)|^2$.

Or

(b) Let H be a Hilbert space and let f be an arbitrary functional in H^*. Prove that there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

20. (a) If P is protection on H with range M and null space N, prove that $M \perp N \Leftrightarrow P$ is self-adjoint and in this case show that $N = M^{\perp}$.

Or

(b) Let T be an arbitrary operator on H. Let $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_m$ be the eigen values and let $M_1, M_2, \ldots\ldots., M_m$ be their corresponding eigen spaces. Prove that if T is normal then the M_i's are pairwise orthogonal and span H.
